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Interference Cancellation by Repeated Filtering in the Fractional Fourier Transform Domain Using Mean-Square Error Convergence

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ABSTRACT
The Fractional Fourier Transform (FrFT) is a useful tool that separates signals-of-interest (SOIs) from interference and noise in non-stationary environments. This requires estimation of the rotational parameter 'a' to rotate the signal to a new domain along an axis 't_a', in which the interference can best be filtered out. The value of 'a' is typically chosen as that which minimizes the mean-square error (MSE) between the desired SOI and its estimate or that minimizes the overlap between the signal and noise, projected onto the axis 't_a'. In this paper, we extend this concept to perform repeated filtering, in multiple FrFT domains to reduce the MSE further than can be done with a single FrFT. We perform this solely using MSE as the metric by which to compute 'a' at each stage, thereby simplifying the approach and improving performance over conventional single stage FrFT methods or methods based solely on the frequency domain filtering, such as the Fast Fourier transform (FFT). We show that the proposed method improves the MSE two or three orders of magnitude over the conventional methods using $L \leq 3$ stages of FrFT filtering.
Keywords **:** Fractional Fourier Transform, Interference Cancellation, Mean-Square Error

I. INTRODUCTION

The Fractional Fourier Transform (FrFT) has a wide range of applications in fields including signal processing, radar, and communications. It is a very effective method for separating a signal-of-interest (SOI) from interference and/or noise in non-stationary signals, which are found in real-world scenarios [8]. The FrFT translates the received signal to an axis in the time-frequency plane where the SOI and interference may be separable [1], when they are not separable in the frequency domain, as given by the conventional Fast Fourier transform (FFT), or in the time domain. This can be visualized using the concept of a Wigner Distribution (WD).

The WD of a signal and interference, as shown in Fig. 1, illustrates how the FrFT may be used to greatly improve signal separation and how repeated filtering may be required. In non-stationary environments, both the SOI x(t) and the interference $x_I(t)$ vary as a function of time and frequency. The WD shows how they both independently vary. Note that they both overlap in the time domain ($t_{a=0}$) and in the frequency domain ($t_{a=1}$), but there are other axes where they do not overlap. In this illustration, two rotations are required to completely filter out the interference. First, we rotate to a_1, $0 < a_1 < 2$, to filter out the blue portion of the interfering signal, and then we rotate to a_2, $0 < a_2 < 2$, to filter out the remaining green part. Hence, by finding the optimum axes and rotating to them using the FrFT, we can filter out the interference completely and achieve significant interference suppression (IS) improvements over conventional time, e.g. minimum mean-square error (MMSE), or frequency (e.g. FFT) filtering. The optimum rotational axes can be found by searching, using an MMSE criterion.

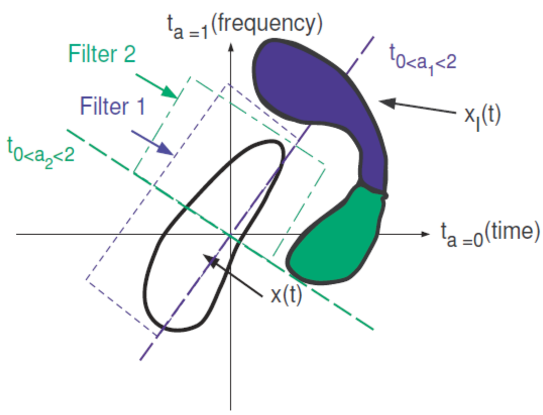

Fig. 1 Wigner Distribution of Signal x(t) and Interference $x_I(t)$ Shows Optimum Axes t_{a_1} and t_{a_2} for Filtering Out Interference

When applying the FrFT to perform interference suppression, we must first estimate the rotational parameter 'a'. Conventional FrFT methods rely on choosing the value of 'a', $0 \leq a \leq 2$, which produces the minimum mean-square error (MMSE) between a desired (training) signal and its estimate [10]. This often results in large errors because typically the sample support in a non-stationary environment is small, but MMSE techniques require lots of samples. One approach to overcome this uses an algorithm that minimizes the overlap of the SOI and interference using the relation between the FrFT and the WD and use reduced rank filtering to cancel the interference [11]. Another approach is to perform repeated filtering in the FrFT domain to cancel interference [4]. This method requires repeated calculation of a series of optimum filter coefficients until MMSE converges.

In this paper, we present a method that simplifies that in [4] by performing the filtering by finding the MMSE as in [10] and repeating the calculation until the MMSE between the SOI estimate and true SOI is less than ϵ, set here to $\epsilon = 10^{-3}$. Hence we improve upon the performance in [10] using a simpler method than given in [4] for repeated FrFT domain filtering. However, because this is still an MMSE based approach, and sample support is limited in a nonstationary environment, performance meeting that in [11] may not be achievable in general.

The paper outline is as follows: Section II briefly reviews the FrFT and its relation to the Wigner Distribution, which is a useful visual tool for the FrFT. Section III presents the signal model. Section IV discusses the conventional FrFT solutions using MMSE and FFT methods, and Section V presents the proposed solution using repeated filtering until the MMSE converges. Section VI has simulation results showing the robust performance of the proposed FrFT method. Conclusions and remarks on future work are given in Section VII.

II. THE FRACTIONAL FOURIER TRANSFORM (FRFT)

The continuous time FrFT and its properties are well-defined in the literature (see for example [8]). In discrete time, we can model the N × 1 FrFT of an N × 1 vector **x** as

$$\mathbf{X}_a = \mathbf{F}^a \mathbf{x}, \tag{1}$$

where $0 < |a| < 2$, $\mathbf{F}^a$ is an N × N matrix whose elements are given by ([3] and [8])

$$\mathbf{F}^a[m,n] = \sum_{k=0, k\neq(N-1+(N)_2)}^{N} u_k[m] e^{-j\frac{\pi}{2}ka} u_k[n], \tag{2}$$

$u_k[m]$ and $u_k[n]$ are the eigenvectors of the matrix **S** [3]

$$\mathbf{S} = \begin{bmatrix} C_0 & 1 & 0 & \dots & 1 \\ 1 & C_1 & 1 & \dots & 0 \\ 0 & 1 & C_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \dots & C_{N-1} \end{bmatrix}, \tag{3}$$

and

$$C_n = 2cos(\frac{2\pi}{N} n) - 4. \tag{4}$$

The Wigner Distribution (WD) is a time-frequency representation of a signal and may be viewed as a generalization of the Fourier Transform, which is solely the frequency representation. It can be shown that the projection of the WD of a signal x(t) onto an axis t_a gives the energy of the signal in the FrFT domain 'a', $|\mathbf{X}_a(t)|^2$ (see e.g. [5] or [6]). In discrete time, the WD of a signal x[n] is written as [7]

$$W_x[\frac{n}{2f_s}, \frac{kf_s}{2N}] = e^{j\frac{\pi}{N}kn} \sum_{m=l_1}^{l_2} x[m]x^*[n-m] e^{j\frac{2\pi}{N}km}, \tag{5}$$

where $l_1 = \max(0, n-(N-1))$ and $l_2 = \min(n, N-1)$.

III. SIGNAL MODEL

Without loss of generality, we model the SOI as a digital, baseband binary phase shift keying (BPSK) signal whose elements are in (−1,+1) that we would like to estimate in the presence of non-stationary interference, and/or a nonstationary channel. The number of samples we process is $N = N_1 SPB$, where we have N_1 bits per block and SPB samples per bit. The SOI is denoted in discrete time, vector form as the N × 1 vector **x**(i). This SOI is corrupted by a non-stationary interferer $\mathbf{x}_I(i)$, to be described in Section VI, and by an additive white Gaussian noise (AWGN) signal **n**(i). There may also be a non-stationary channel **h**(i) that is corrupting the SOI. Here, index i denotes the i^{th} sample, where i = 1, 2, ,…, N. The received signal **y**(i) is then

$$\mathbf{y}(i) = \mathbf{x}(i) * \mathbf{h}(i) + \mathbf{x}_I(i) + \mathbf{n}(i), \tag{6}$$

where '*' denotes convolution. We obtain an estimate of the transmitted signal **x**(i), denoted $\hat{\mathbf{x}}(i)$, by first transforming the received signal to the FrFT domain, applying an adaptive filter, and taking the inverse FrFT. This is written as [10]

$$\hat{\mathbf{x}}(i) = \mathbf{F}^{-a} \mathbf{G} \mathbf{F}^a \mathbf{y}(i), \tag{7}$$

where $\mathbf{F}^a$ and $\mathbf{F}^{-a}$ are the N × N FrFT and inverse FrFT matrices of order 'a', respectively, and

$$\mathbf{g} = diag(\mathbf{G}) = (g_0, g_1, g_{N-1}) \tag{8}$$

is an N × 1 set of optimum filter coefficients to be found. The notation diag($\mathbf{G}$) = (g_0, g_1, .., g_{N-1}) means that matrix $\mathbf{G}$ has the scalar coefficients g_0, g_1, ..., and g_{N-1} as its diagonal elements, with all other elements equal to zero.

IV. CONVENTIONAL FRFT METHODS

A. MMSE-FrFT Method

Conventional MMSE-based FrFT methods are well-known and their description is simply repeated here for completeness. MMSE-FrFT techniques seek to minimize the error between the desired signal $\mathbf{x}$(i) and its estimate $\hat{\mathbf{x}}$(i). That is, we minimize the cost function

$$J(\mathbf{g}) = \frac{1}{M}\sum_{i=1}^{M} ||\mathbf{F}^{-a}\mathbf{G}\mathbf{F}^{a}\mathbf{y}(i) - \mathbf{x}(i)||^2, \tag{9}$$

The optimum set of filter coefficients $\mathbf{g}_0$ that minimizes the cost function in Eq. (9) can be obtained by computing the partial derivative of the cost function and setting it to zero [10]. That is, $\mathbf{g}_0$ is such that

$$\frac{\partial J(\mathbf{g})}{\partial \mathbf{g}}|_{\mathbf{g}=\mathbf{g}_0} = 0. \tag{10}$$

This is the MMSE-FrFT solution, given by [10]

$$\mathbf{g}_{0,MMSE-FrFT}(i) = \frac{1}{2}\mathbf{Q}^{-1}(i)\mathbf{b}(i), \tag{11}$$

where

$$\mathbf{Q}(i) = (\mathbf{F}^{-a}\mathbf{Z}(i))^H(\mathbf{F}^{-a}\mathbf{Z}(i)), \tag{12}$$

$$\mathbf{z}(i) = [z_0(i)\ z_1(i)\ ...\ z_{N-1}(i)]^T = diag(\mathbf{Z}(i)) = \mathbf{F}^a\mathbf{y}(i), \tag{13}$$

$$\mathbf{b}(i) = (-2Re(\mathbf{x}^H(i)\mathbf{F}^{-a}\mathbf{Z}(i)))^T, \tag{14}$$

and $(\cdot)^H$ denotes Hermitian transpose. We thus choose the value of 'a' as that which minimizes the cost function in Eq. (9). We point out that we must compute the cost function over the range of 'a' from 0 < a < 2 by first computing $\mathbf{g}_{0,\mathrm{MMSE-FrFT}}$ from Eq. (11), to find the best value of 'a'. Note also that this solution requires a training sequence, $\mathbf{x}$(i).

B. MMSE-FFT Method

The MMSE-FFT solution is obtained by setting a = 1 in calculating $\mathbf{g}_0$ from Eqs. (11)–(14) since $\mathbf{F}^1$ reduces to the standard FFT. So, we can write

$$\mathbf{g}_{0,MMSE-FFT}(i) = \mathbf{g}_{0,MMSE-FrFT}(i)|_{a=1}. \tag{15}$$

V. PROPOSED ALGORITHM

The proposed algorithm computes the signal estimate $\hat{\mathbf{x}}$(i) using Eq. (11) and Eq. (7) and chooses the value of 'a' for which the MSE between $\hat{\mathbf{x}}$(i) and $\mathbf{x}$(i) is minimum. The search is performed for 0 ≤ a < 2 using a step size of Δa = 0.01. We then repeat the calculation by using an updated version of the received signal obtained from the previous estimate. That is, we set $\mathbf{y}$(i) = $\hat{\mathbf{x}}$(i) and again calculate the MSE by searching for the best value of 'a' for which the MSE is minimized. This could yield a different value of 'a' than in the previous step. These steps are repeated for up to L iterations until the MSE drops below the threshold ϵ or reaches a minimum value. The algorithm is summarized in Table I. Note that l = 1 is the conventional MMSE-FrFT solution, and l = 1, a = 1 is the conventional MMSE-FFT solution, which we compare to the proposed approach in Section VI.

Note that the values a(l) computed by the above algorithm would in practice be used to filter the data present in the signal following the training sequence. As the statistics of the environment and channel change, a new training sequence could be used to update the a(l)'s. In the next section, we show simulation results, which indicate that typically very few iterations, i.e. L ≤ 3 are needed to reduce the MSE by more than an order of magnitude over the single iteration algorithm of [10] when both dynamic interference and a time-varying channel are present.

TABLE I. Proposed Repeated Filtering Algorithm

Initialize: $l = 1$; % Iteration $l = 1, 2, ..., L$
$\mathbf{y}(i,l) = \mathbf{y}(i)$; % Received signal
1. % Compute MMSE filter coefficients for $0 \leq a(l) < 2$
$\mathbf{g}_{0,MMSE-FrFT}(i,l) = \frac{1}{2}\mathbf{Q}^{-1}(i)\mathbf{b}(i)$;
where
$\mathbf{Q}(i,l) = (\mathbf{F}^{-a(l)}\mathbf{Z}(i))^H(\mathbf{F}^{-a(l)}\mathbf{Z}(i,l))$;
and
$\mathbf{z}(i,l) = diag(\mathbf{Z}(i,l)) = \mathbf{F}^{a(l)}\mathbf{y}(i,l)$;
2. % Compute the best 'a' which minimizes the MSE and the MSE
MSE-FrFT(l) = $\underset{a(l)}{arg\ min}\ \frac{1}{M}\sum_{i=1}^{M} \lVert\hat{\mathbf{x}}(i,l) - \mathbf{x}(i)\rVert^2$;
where
$\hat{\mathbf{x}}(i,l) = \mathbf{F}^{-a(l)}\mathbf{G}(l)\mathbf{F}^{a(l)}\mathbf{y}(i,l)$;
3. % Update the received signal using the new estimate
$\mathbf{y}(i,l+1) = \hat{\mathbf{x}}(i,l)$;
4. % Increment $l = l + 1$ for the next iteration.
5. Repeat Steps $1 - 4$ until MSE-FrFT(l) $< \epsilon$.

VI. SIMULATIONS

We assume the SOI $\mathbf{x}$(i) is a BPSK signal as discussed above and let N_1 = 10 bits per block, and SPB = 4 samples per bit, so that N = 40 samples. In the first example, we model the channel as a time-varying, bandpass signal whose center frequency is changing with time as [5]

$$h(i) = e^{-j2\pi(i/f_s)^2} sinc(i/f_s), \tag{16}$$

where $f_s = SPB \cdot R_b$ is the sampling rate, and R_b is the bit rate. We let the interferer be a chirp signal given by

$$\mathbf{x}_I(i) = A_I e^{-j\pi(i/f_s)^2}, \tag{17}$$

where the amplitude of the chirp, A_I, is the strength of the interfering signal, obtained by setting its mean amplitude based upon a desired carrier-to-interference ratio (CIR), and we set the amplitude of the AWGN based upon a desired E_b/N_0. Specifically, we set the amplitude of the SOI to $A = \frac{1}{\sqrt{SPB}}$ and set the amplitude of the interferer to $A_I = 10^{-CIR/20}$, where the CIR is given in dB; note a negative CIR means that the interferer is stronger than the SOI. We further set the amplitude of the AWGN to be $\sigma_N = \sqrt{\frac{1}{2 \cdot 10^{\frac{E_b/N_0}{10}}}}$.

The transmitted signal $\mathbf{x}(i)$, the received signal $\mathbf{y}(i)$, the signal estimates after l = 1 iteration (i.e. the MMSE-FrFT estimate in [10]) and l = 2 iterations (the proposed solution), and the l = 1, a = 1 (FFT estimate) signal are all shown for comparison in Fig. 2. Note that the algorithm is able to accurately estimate the signal after just two iterations, with MSE-FrFT(1) = 0.0089, and MSE-FrFT(2) = $4.0259 \cdot 10^{-5}$, whereas MSE-FFT = 0.1031. The best ‘a’ at each iteration is shown in the figure.

In the second example, we let CIR = 0 dB, with the result shown in Fig. 3. With the very low CIR, we expect to see some performance degradation, and this can somewhat be corrected by adding a third iteration, but because there is some signal overlap in the time and frequency domains, we cannot achieve perfect cancellation. However, we do see a reduction in MSE. The MSEs are MSE-FFT = 0.2276, MSE-FrFT(1) = 0.0096, MSE-FrFT(2) = 0.0061, and MSE-FrFT(3) = 0.0022.

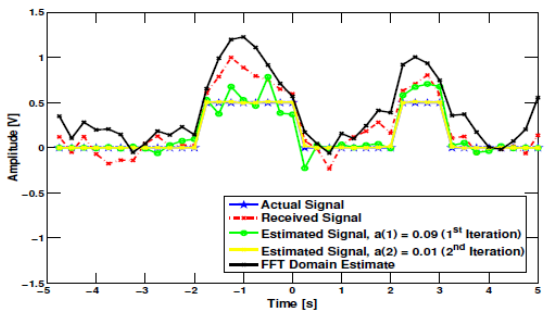

Fig. 2 Chirp Interferer; E_b/N_0 = 10 dB, CIR = 5 dB

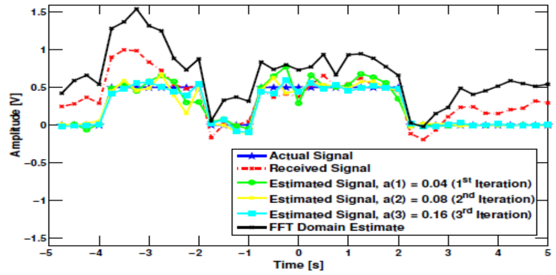

Fig. 3 Chirp Interferer; E_b/N_0 = 10 dB, CIR = 0 dB

The third example uses an interferer in the form of a Gaussian pulse, given by

$$\mathbf{x}_I(i) = A_I \beta e^{-\pi(i/f_s - \phi)^2}, \tag{18}$$

where β and ϕ are the amplitude and phase of the pulse, respectively, uniformly distributed in (0.5,1.5), and we set the CIR to 5 dB. All other parameters are the same as before. The plot is shown in Fig. 4. The errors are calculated as MSE-FFT = 0.1259, MSE-FrFT(1) = 0.01, and MSEFrFT(2) = 0.001. Only two iterations are needed as the CIR is high enough in this case.

In the last example, shown in Fig. 5, we repeat the above but with CIR = 0 dB. Now three iterations are needed, and we obtain MSE-FFT = 0.013, MSE-FrFT(1) = 0.004, MSEFrFT(2) = 0.0023, and MSE-FrFT(3) = $2.0183 \cdot 10^{-4}$. Hence, in these last two examples, better interference cancellation is achievable due to the nature of the interference, but when CIR is reduced, we require three stages of filtering versus two.

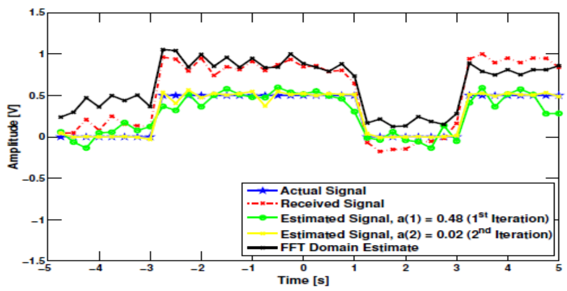

Fig. 4 Gaussian Pulse Interferer; E_b/N_0 = 10 dB, CIR = 5 dB

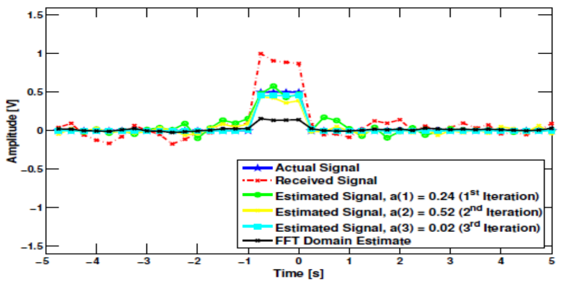

Fig. 5 Gaussian Pulse Interferer; E_b/N_0 = 10 dB, CIR = 0 dB

VII. CONCLUSION

This paper presents a simple algorithm that uses repeated filtering in FrFT domains to improve upon signal demodulation in non-stationary interference. Mean-square error (MSE) convergence is the metric chosen to determine how many iterations are required, where the signal estimate at each iteration is used to update the next iteration. We demonstrate that this outperforms conventional methods that use either a single MSE estimate or an FFT approach, with two or three orders of magnitude improvement in MSE after just two or three iterations, and the algorithm is simpler to implement than other techniques that rely on iterating on optimum filtering coefficients. Future work includes applying the technique to other types of real-world signals in communications applications.

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